Lecture 14
Minimum Spanning Trees (MSTs): Prim, Kruskal

CS 161 Design and Analysis of Algorithms

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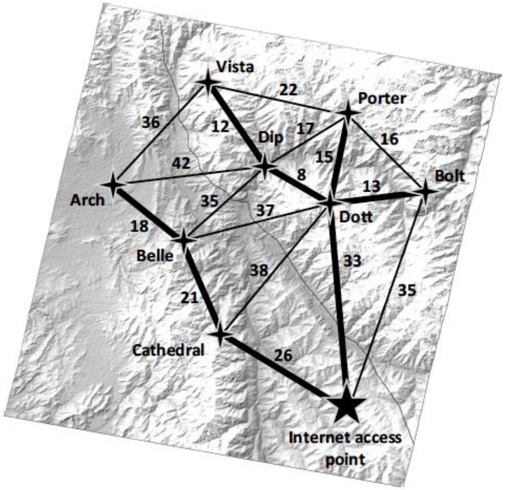
Application: Connecting a Network

- Suppose the remote mountain country of Vectoria has been given a major grant to install a large Wi-Fi the center of each of its mountain villages.
- Communication cables can run from the main Internet access point to a village tower and cables can also run between pairs of towers.
- The challenge is to interconnect all the towers and the Internet access point as **cheaply** as possible.

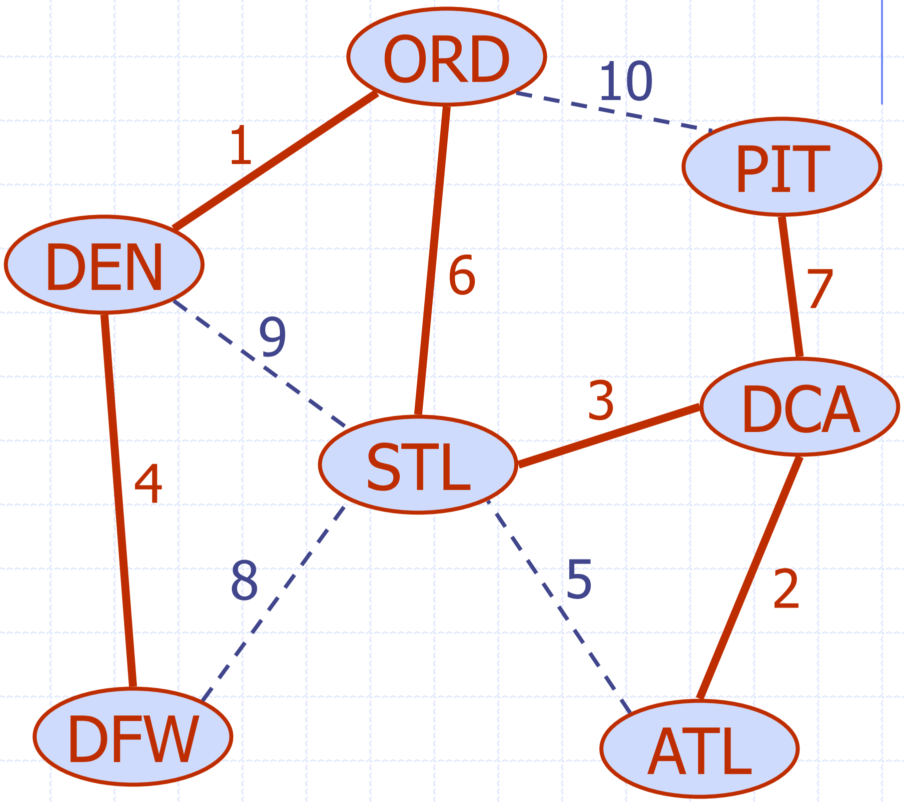


Application: Connecting a Network

- We can model this problem using a graph, **G**, where each vertex in **G** is the location of a Wi-Fi the Internet access point, and an edge in **G** is a possible cable we could run between two such vertices.
- Each edge in **G** could then be given a weight that is equal to the cost of running the cable that that edge represents.
- Thus, we are interested in finding a connected acyclic subgraph of **G** that includes all the vertices of **G** and has minimum total cost.
- Using the language of graph theory, we are interested in finding a **minimum spanning tree (MST)** of **G**.

Minimum Spanning Trees

- Spanning subgraph
  - Subgraph of a graph ***G*** containing all the vertices of ***G***
- Spanning tree
  - Spanning subgraph that is itself a (free) tree
- Minimum spanning tree (MST)
  - Spanning tree of a weighted graph with minimum total edge weight
- Applications
  - Communications networks
  - Transportation networks



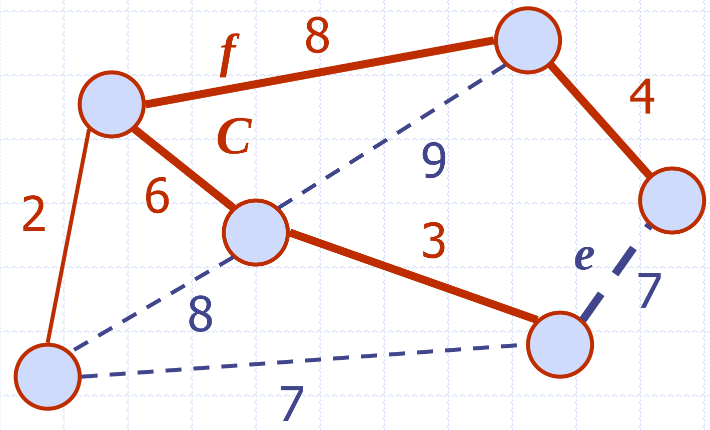
Cycle Property

Cycle Property:

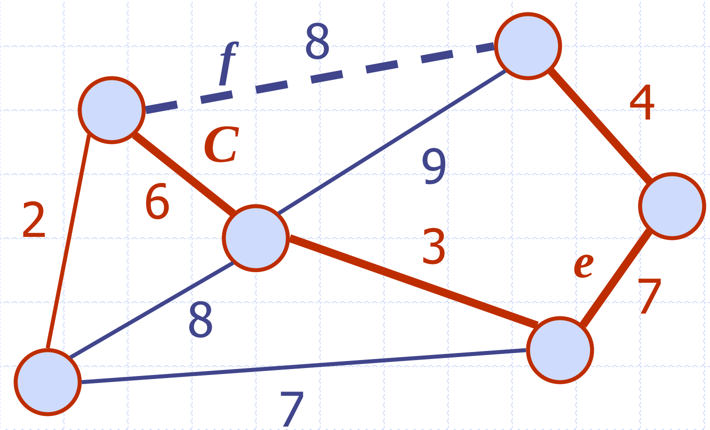
- Let T be a minimum spanning tree of a weighted graph G
- Let e be an edge of G that is not in T and C let be the cycle formed by e with T
- For every edge f of C, $weight(f) \leq weight(e)$

Proof:

- By contradiction
- If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing e with f



Replacing f with e yields a better spanning tree



Partition Property

Partition Property:

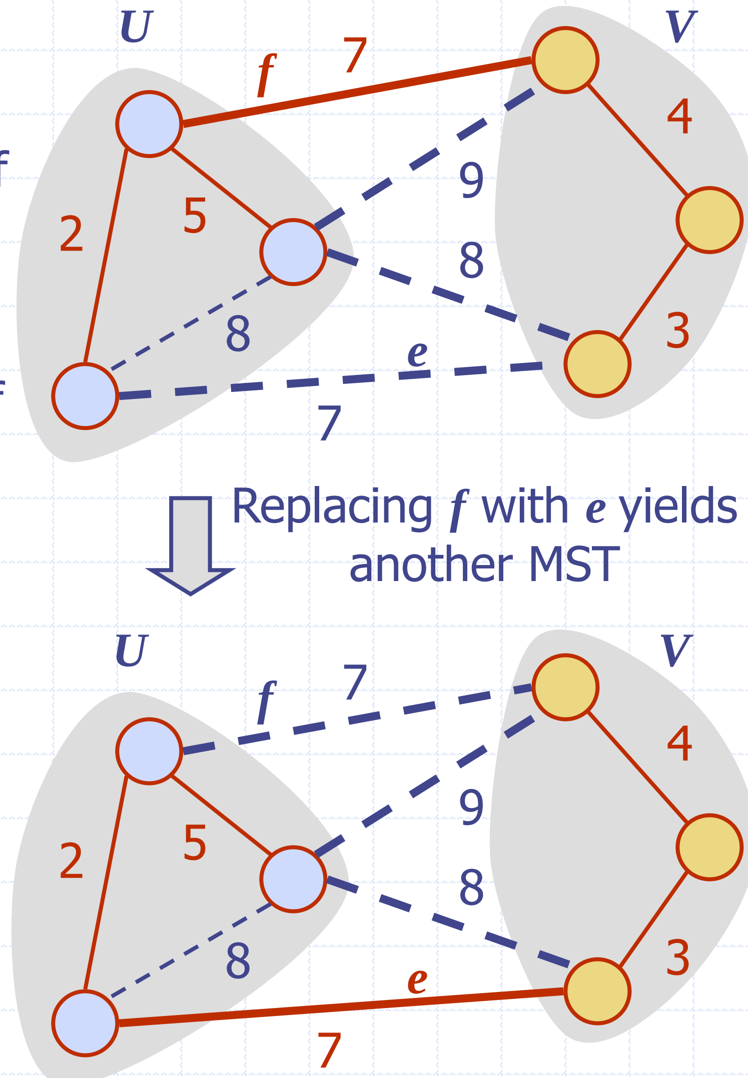
- Consider a partition of the vertices of G into subsets U and V
- Let e be an edge of minimum weight across the partition
- There is a minimum spanning tree of G containing edge e

Proof:

- Let T be an MST of G
- If T does not contain e, consider the cycle C formed by e with T and let f be an edge of C across the partition
- By the cycle property,

$$weight(f) \leq weight(e)$$

- Thus, $weight(f) = weight(e)$
- We obtain another MST by replacing f with e



Prim-Jarnik's Algorithm

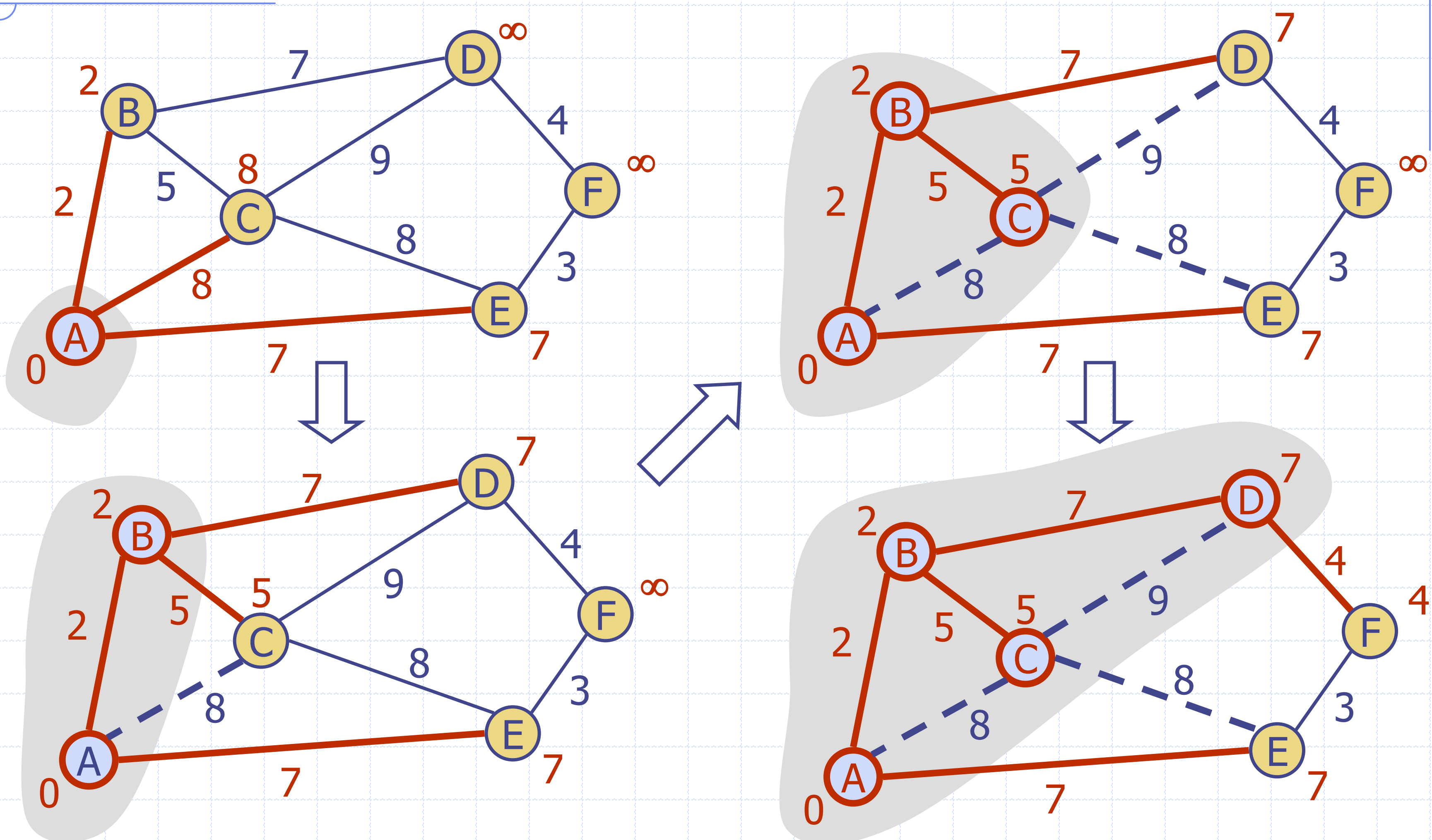
- Similar to Dijkstra's algorithm
- We pick an arbitrary vertex s and we grow the MST as a cloud of vertices, starting from s
- We store with each vertex v label $d(v)$ representing the smallest weight of an edge connecting v to a vertex in the cloud
- At each step:
  - We add to the cloud the vertex u outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to u

Prim-Jarnik Pseudo-code

Algorithm PrimJarníkMST(G):
  Input: A weighted connected graph G with n vertices and m edges
  Output: A minimum spanning tree T for G

```
Pick any vertex v of G
D[v] ← 0
for each vertex u ≠ v do
    D[u] ← +∞
Initialize T ← ∅.
Initialize a priority queue Q with an item ((u, null), D[u]) for each vertex u,
where (u, null) is the element and D[u] is the key.
while Q is not empty do
    (u, e) ← Q.removeMin()
    Add vertex u and edge e to T.
    for each vertex z adjacent to u such that z is in Q do
        // perform the relaxation procedure on edge (u, z)
        if w((u, z)) < D[z] then
            D[z] ← w((u, z))
            Change to (z, (u, z)) the element of vertex z in Q.
            Change to D[z] the key of vertex z in Q.
return the tree T
```

Example



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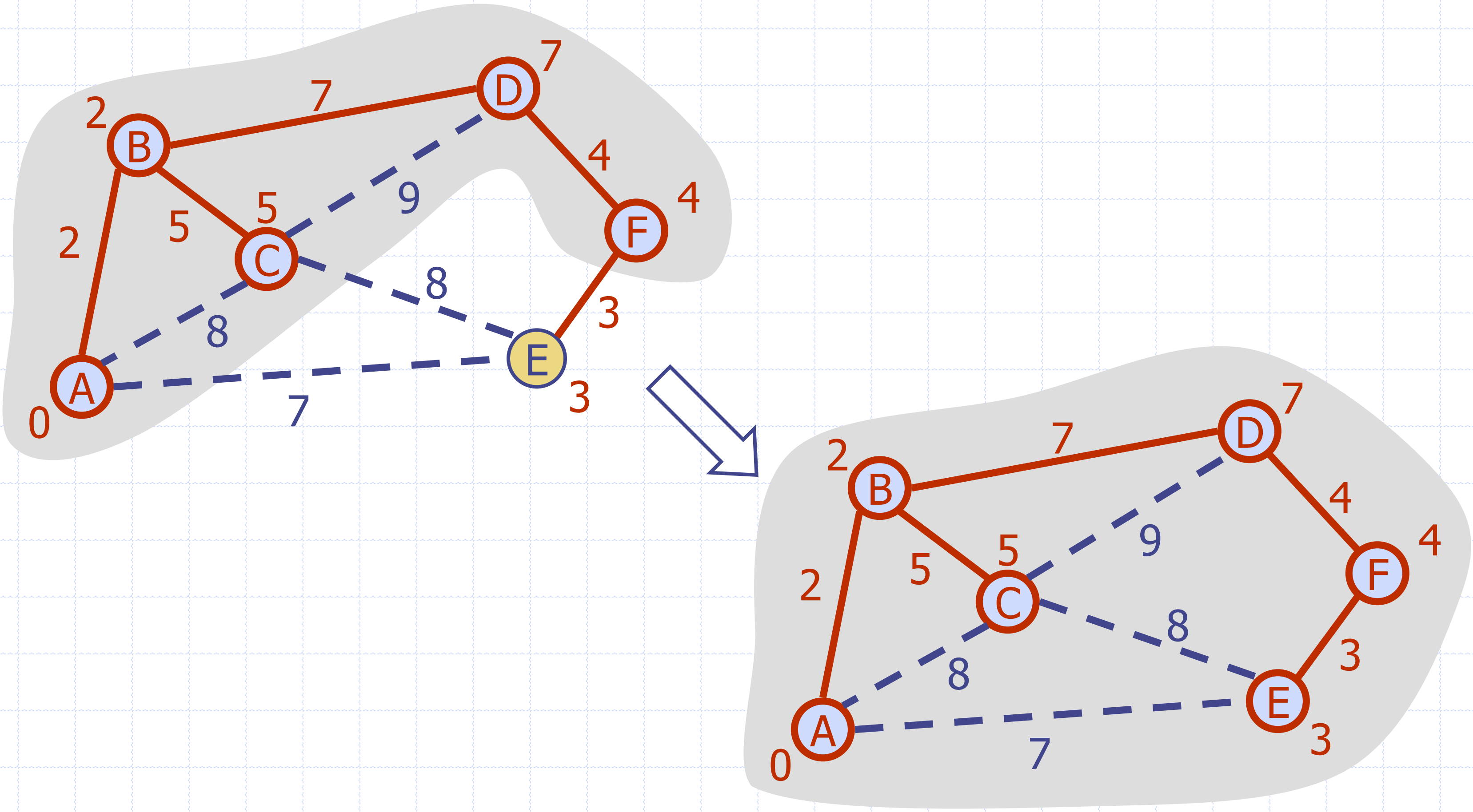
Example (contd.)



Analysis

- Graph operations
  - We cycle through the incident edges once for each vertex
- Label operations
  - We set/get the distance, parent and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex w in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Prim-Jarnik's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$
- The running time is $O(m \log n)$ since the graph is connected

Kruskal's Approach

- Maintain a partition of the vertices into clusters
  - Initially, single-vertex clusters
  - Keep an MST for each cluster
  - Merge "closest" clusters and their MSTs
- A priority queue stores the edges outside clusters (or you could even sort the edges)
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - One cluster and one MST

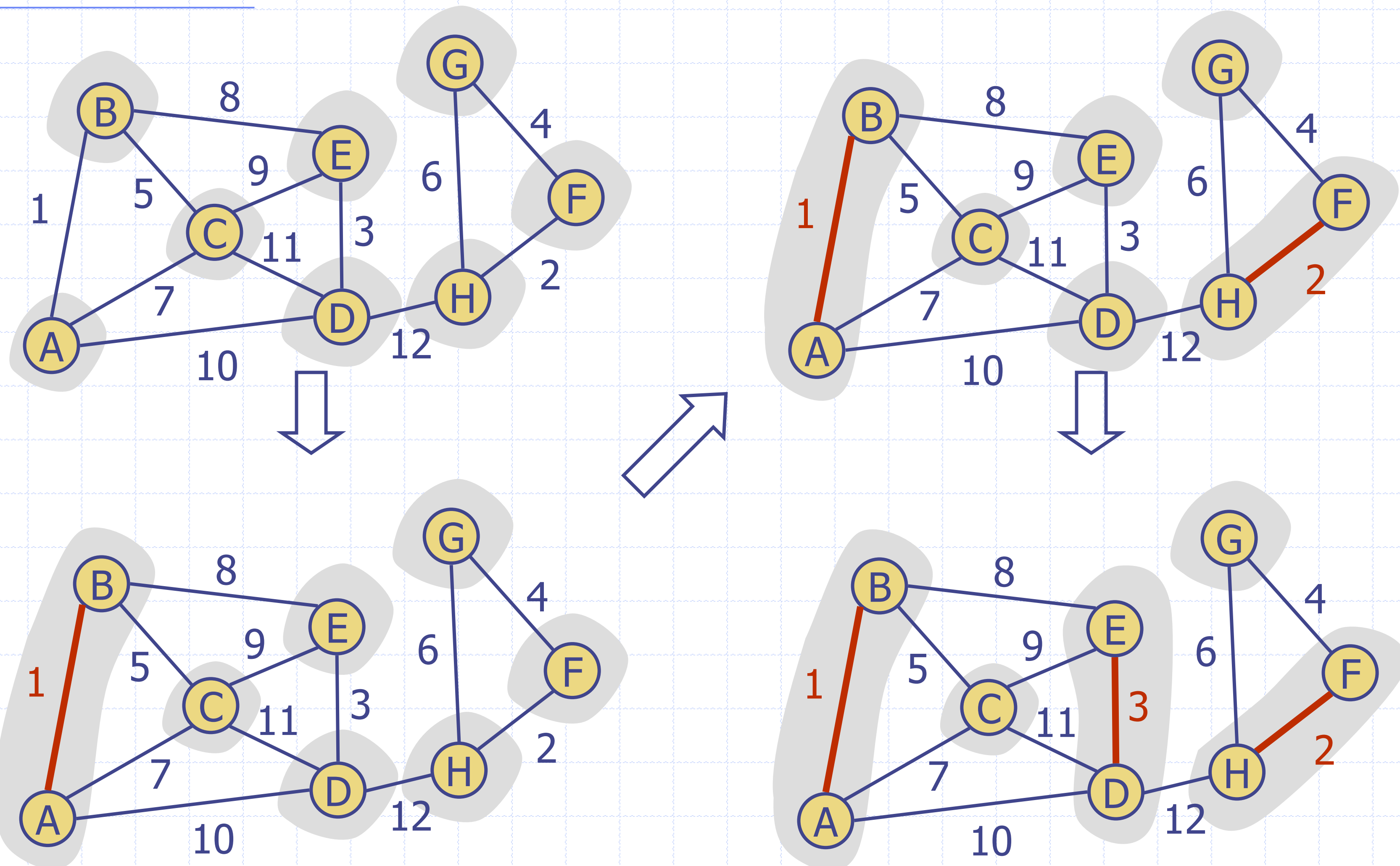
Kruskal's Algorithm

Algorithm KruskalMST(G):

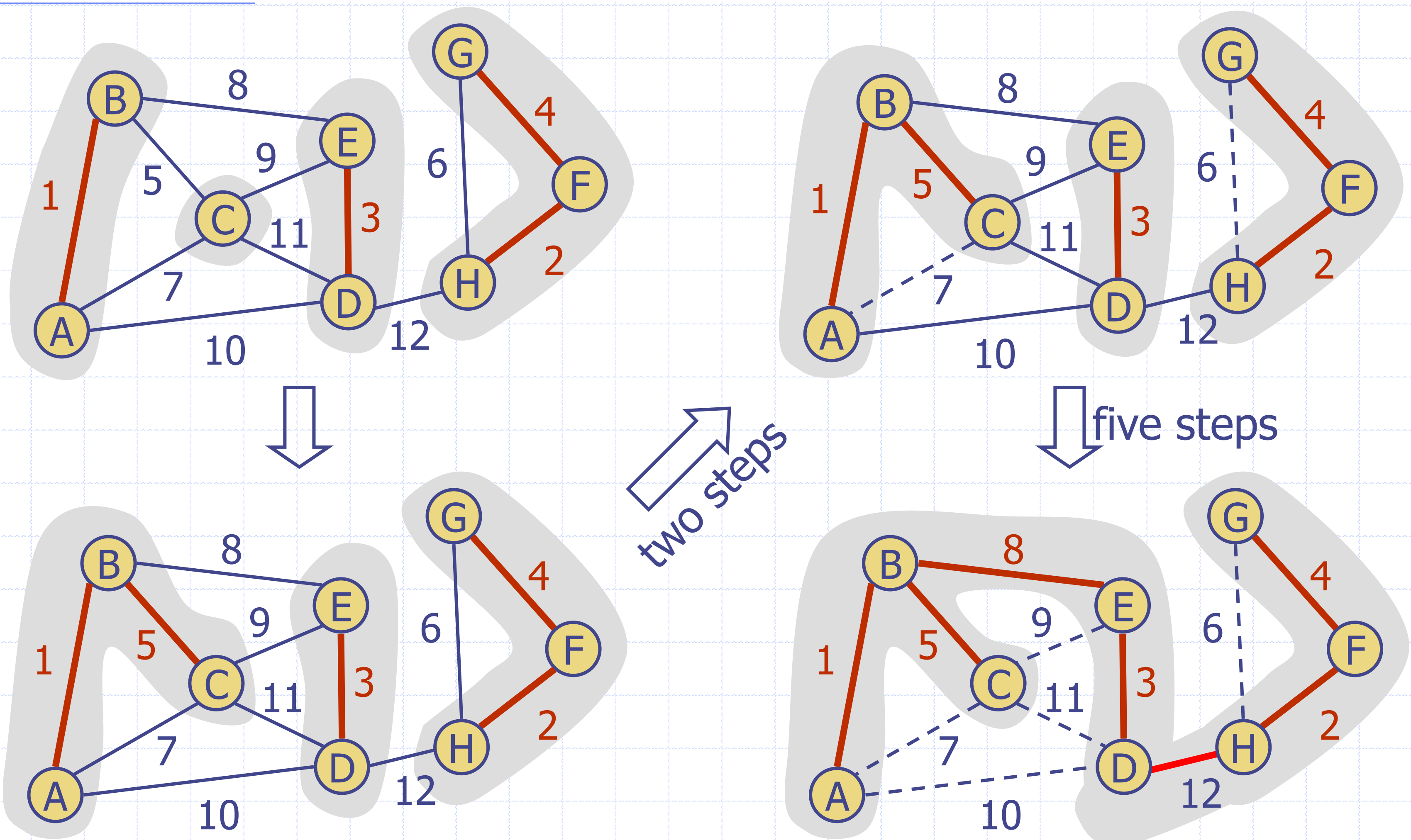
Input: A simple connected weighted graph G with n vertices and m edges
Output: A minimum spanning tree T for G

```
for each vertex v in G do
    Define an elementary cluster C(v) ← {v}.
Let Q be a priority queue storing the edges in G, using edge weights as keys
T ← ∅        // T will ultimately contain the edges of the MST
while T has fewer than n − 1 edges do
    (u, v) ← Q.removeMin()
    Let C(v) be the cluster containing v
    Let C(u) be the cluster containing u
    if C(v) ≠ C(u) then
        Add edge (v, u) to T
        Merge C(v) and C(u) into one cluster, that is, union C(v) and C(u)
return tree T
```

Example of Kruskal’s Algorithm



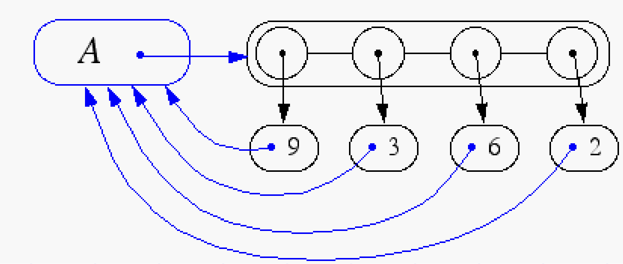
Example (contd.)



Data Structure for Kruskal's Algorithm

- The algorithm maintains a forest of trees
- A priority queue extracts the edges by increasing weight
- An edge is accepted it if connects distinct trees
- We need a data structure that maintains a partition, i.e., a collection of disjoint sets, with operations:
  - makeSet(u): create a set consisting of u
  - find(u): return the set storing u
  - union(A, B): replace sets A and B with their union

List-based Partition



- Each set is stored in a sequence
- Each element has a reference back to the set
  - operation find(u) takes O(1) time, and returns the set of which u is a member.
  - in operation union(A,B), we move the elements of the smaller set to the sequence of the larger set and update their references
  - the time for operation union(A,B) is min(|A|, |B|)
- Whenever an element is processed, it goes into a set of size at least double, hence each element is processed at most log n times

Partition-Based Implementation

- Partition-based version of Kruskal's Algorithm
  - Cluster merges as unions
  - Cluster locations as finds
- Running time $O((n + m) \log n)$
  - Priority Queue operations: $O(m \log n)$
  - Union-Find operations: $O(n \log n)$